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DISCUSSION OF STEADY-STATE FORCED VIBRATION OF CONTINUOUS FRAMES

(Published in June, 1952)

By E. F. Masur, A. S. Veletsos, William A. Conwell, and C. T. Looney

STRUCTURAL DIVISION

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DISCUSSION

E. F. MASUR[2].—Except for a few relatively simple types of continuous structures, formidable numerical difficulties are involved in the determination of the natural frequencies and modes of vibration of continuous frames and the response of these structures to applied periodic forces. The author should be commended for applying successive approximation methods in the solution of such problems because most engineers are now conversant with such methods as far as problems of statics are concerned. However, as presented, the solution ignores the transient response, although the critical deflections and stresses, which are of particular interest to the engineer, are often greatly affected by transient vibrations, especially in the case of slight damping. The author alludes to this in the last paragraph of the "Summary"; however, it is not clear to the writer how the author proposes to apply his method to the determination of the natural frequencies and modes of the continuous structure.

It is the writer's opinion that a great deal of computing labor can be saved in finding the response of a simple beam by avoiding the use of infinite series. This suggestion is particularly true for the calculation of the stiffness factor and the carry-over factor (see Eq. 5*b*). In what follows, it will be assumed that the beam is undamped, although the effect of damping can be included with little additional labor.

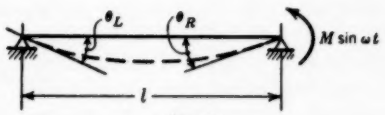

FIG. 9.

If the beam shown in Fig. 9 is subjected to a periodic couple at the right end as shown, the deflection $y(x, t)$ satisfies the differential equation:

$$E I \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = 0 \quad \ldots\ldots (33)$$

letting the steady-state solution be of the type—

$$y(x, t) = Y(x) \sin \omega t = Y(x) \sin 2 \pi f t \quad \ldots\ldots (34)$$

in which $Y(x)$ denotes the mode of vibration, and considering the boundary conditions,

$$y(0, t) = \left.\frac{\partial^2 y}{\partial x^2}\right|_{(0,\ t)} = y(l, t) = 0 \quad \ldots\ldots (35a)$$

$$E I \left.\frac{\partial^2 y}{\partial x^2}\right|_{(l,\ t)} = -M \sin \omega t \quad \ldots\ldots (35b)$$

the deflection function is obtained

$$Y(x) = \frac{M l^2}{2 E I} \frac{1}{\phi^2} \left(\frac{\sin \frac{\phi x}{l}}{\sin \phi} - \frac{\sinh \frac{\phi x}{l}}{\sinh \phi} \right) \quad \ldots\ldots (36)$$

NOTE.—This paper by C. T. G. Looney was published in June, 1952, as *Proceedings-Separate No. 136*. The numbering of footnotes and equations in this Separate is a continuation of the consecutive numbering used in the original paper.

[2] Associate Prof. of Civ. Eng., Illinois Inst. of Technology, Chicago, Ill.

in which $\phi = \sqrt[4]{\frac{m\,\omega^2\,l^4}{E\,I}} = \pi\sqrt{\frac{\omega}{\omega_1}} = \pi\sqrt{\frac{f}{f_1}}$, with ω_1, representing the fundamental frequency of the simple beam, so that $\omega_1 = \sqrt{\frac{\pi^4\,E\,I}{m\,l^4}} = 2\,\pi\,f_1$. By differentiation, it follows from Eq. 36 that—

$$\theta_L = \left.\frac{dY}{dx}\right|_{(0)} = \frac{M\,l}{2\,E\,I\,\phi}\,(\csc\phi - \operatorname{csch}\phi) \ldots\ldots\ldots\ldots (37a)$$

and that—

$$\theta_R = -\left.\frac{dY}{dx}\right|_{(l)} = \frac{M\,l}{2\,E\,I\,\phi}\,(\coth\phi - \cot\phi) \ldots\ldots\ldots\ldots (37b)$$

Introducing the notation,

$$s = \tfrac{1}{2}\,(s' + s'') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38a)$$

and

$$c = \tfrac{1}{2}\,(c' + c'') \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38b)$$

in which $s' = \frac{1}{\phi^2}\,(\phi\csc\phi - 1)$, $s'' = \frac{1}{\phi^2}\,(1 - \phi\operatorname{csch}\phi)$, $c' = \frac{1}{\phi^2}\,(1 - \phi\cot\phi)$, and $c'' = \frac{1}{\phi^2}\,(\phi\coth\phi - 1)$, Eqs. 37 can be written in the form:

$$\theta_L = \frac{M\,l}{E\,I}\,s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39a)$$

$$\theta_R = \frac{M\,l}{E\,I}\,c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39b)$$

The expressions, $\frac{E\,I}{l}\,\frac{1}{c}$ and $\frac{s}{c}$, represent, respectively, the stiffness and carry-over factors for the beam.

The quantities c', c'', s', and s'' are identical with the coefficients of the moments appearing in the familiar four-moment equations of continuous beam-columns; the single primes refer to members in compression and the double primes to those in tension. These coefficients have been tabulated and are available in several textbooks, such as the one by the late Friedrich Bleich,[3] M. ASCE. Thus, the problem at hand has been reduced to a similar and more familiar one of analyzing a structure consisting of continuous beam-columns, in which the coefficients of the moments in the continuity equations are the mean between the coefficients for tension members and those for compression members.

In closing, the writer would like to point out that the problem under consideration, for the undamped case, has been treated previously in a similar manner.[4] In that paper, the steady-state response of a continuous structure was obtained by both the successive removal of angular discontinuities and moment distribution. It was found that convergence of the latter process is assured only when the applied frequency is less than the fundamental frequency

[3] "Buckling Strength of Metal Structures," by Friedrich Bleich, McGraw-Hill Book Co., Inc., New York, N. Y., 1952, p. 204.

[4] "On Moment Balancing in Structural Dynamics," by R. E. Gaskell, *Quarterly of Applied Mathematics*, Vol. 1, 1943, pp. 237–249.

of the continuous structure. To the writer's knowledge, the convergence of the method of removing angular discontinuities has not been investigated analytically; however, it is apparent that its range of convergence is also limited.

A. S. VELETSOS[5].—An extension of the method of L. E. Grinter, M. ASCE, of balancing end angle changes[6] has been presented by the author for the determination of the steady-state response of continuous beams and frames subjected to harmonically varying forces, such as those resulting from rotating machinery. This paper is of considerable significance in that it brings to the attention of the profession an important engineering problem which, unfortunately, has not been discussed sufficiently in the engineering literature of the United States.

The problem considered in this paper has been treated by a number of investigators in Europe. The pioneering work in this field was done by W. L. Cowley and H. Levy[7] who, as early as 1918, investigated the dynamic response of continuous beams. The first practical method for analyzing the steady-state forced vibrations of continuous beams and frames was presented in 1930 by W. Prager,[8] M. ASCE. Mr. Prager developed equations analogous to the three-moment and four-moment equations and applied them in the analysis of several practical cases. To facilitate the application of this method, S. Gradstein and Mr. Prager[9] later published a tabulation of the various coefficients appearing in the three-moment and four-moment equations. A detailed description of the results of these and of other investigations is to be found in a book by K. Hohenemser and Mr. Prager[10]. The extension to the dynamical problem of the modern distribution procedures of indeterminate stress analysis was first presented in 1943 by R. E. Gaskell,[4] who generalized the concepts of stiffness and carry-over factors to include the dynamic effect, developed the appropriate analytical expressions in terms of the functions tabulated by Messrs. Hohenemser and Prager [10] and showed the application of both the method of moment distribution developed by Hardy Cross,[11] Hon. M. ASCE, and the method of distributing angle changes of Mr. Grinter. In this, as in all previous investigations, the effect of damping on the steady-state forced vibration was assumed to be negligible. It is indeed regrettable that the author apparently was not aware of the work of these investigators.

The two distinguishing features of the author's work are: (*a*) The effect of damping, which was neglected in previous investigations, was considered;

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[5] Research Associate in Civ. Eng., Univ. of Illinois, Urbana, Ill.

[6] "Analysis of Continuous Frames by Balancing Angle Changes," by L. E. Grinter, *Transactions*, ASCE, Vol. 102, 1937, pp. 1020-1036.

[7] "Vibration and Strength of Struts and Continuous Beams Under End Thrusts," by W. L. Cowley and H. Levy, *Proceedings*, Royal Soc. of London, Vol. 95, 1918, pp. 440–457.

[8] "Die Beanspruchung von Tragwerken durch schwingende Lasten," by W. Prager, *Ingenieur-Archiv*, Vol. 1, 1930, pp. 527–532.

[9] "Beanspruchung und Formanderung von Stabwerken bei erzwungenen Schwingungen," by S. Gradstein and W. Prager, *ibid.*, Vol. 11, 1932, pp. 622–650.

[10] "Dynamik der Stabwerke," by K. Hohenemser and W. Prager, Julius Springer, Berlin, Germany, 1933, Chapter IV, pp. 238–289.

[11] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 96, 1932, pp. 1–10.

and (*b*) the solution of the governing differential equation was obtained in an infinite series form, whereas in previous studies it was obtained in a closed form. In the steady-state response of continuous systems, damping seems to be of importance only for frequencies of vibration equal to, or very nearly equal to, one of the natural frequencies of the system. The effect of damping at these frequencies is to prevent the stresses and the deflections from attaining infinitely large values. For exciting frequencies sufficiently removed from the critical frequencies of the system, the effect of damping is fairly small; and for all practical cases it can be safely disregarded.

Two of the principal reasons why the Cross method of distributing fixed-end moments and the alternate Grinter method of distributing end angle changes have found such widespread application in the analysis of frames subjected to static loads are as follows: (*a*) These methods, if carried through a sufficiently large number of cycles, will always converge to a unique and correct answer, and (*b*) the rate of convergence of these methods is fairly rapid so that, in general, a small number of distributions will yield answers of satisfactory engineering accuracy. It is to be emphasized that this statement is concerned with static loads. Unfortunately, when applied to the analysis of the steady-state response of systems subjected to pulsating forces, the Cross and Grinter methods will not always converge to the true answer; and for those cases for which the successive approximation procedures do converge, their rate of convergence may be extremely slow. It is beyond the scope of this discussion to define the conditions under which the Cross and Grinter methods will converge when extended to the analysis of systems subjected to pulsating loads. It is sufficient to state that Mr. Gaskell has proved that, for an undamped system, convergence of these procedures can be insured only if the frequency of the pulsating force is below the fundamental or first natural frequency of the system. Under certain conditions, these methods may converge for frequencies of vibration above the first natural frequency of the system; however, lack of convergence is a definite possibility which should not be overlooked.

The author is aware of the possibility of slow convergence, and in the "Summary" he points out the desirability of using modified stiffnesses. It is the opinion of the writer that, except for frequencies of vibration that are below the gravest natural frequency of the system, the use of those methods of indeterminate stress analysis which do not involve problems of convergence is in general more efficient than any of the distribution procedures previously referred to. To those for whom modern methods of analysis are more appealing than the use of the three-moment equation, the method of direct moment distribution[12] of T. Y. Lin, M. ASCE, provides a very efficient alternative. One could also develop an alternate method of direct angle-change distribution. Of course, these methods are effective only for structures that do not involve closed figures. For frames, in general, one must use other more appropriate techniques. Because the choice of an analytical procedure is, to a large extent, a question of personal preference, it does not seem desirable to elaborate further on this point. It is important to state, however, that the most ap-

[12] "A Direct Method of Moment Distribution," by T. Y. Lin, *Transactions*, ASCE, Vol. 102, 1937, pp. 561–571.

parent need at the present stage of engineering knowledge is a detailed tabulation of such quantities as dynamic fixed-end moments or free end slopes, dynamic stiffnesses, and dynamic carry-over factors. In a dynamical problem, the laws of statics alone are not sufficient for evaluating the moments and shears at the interior of a span from the known moments and shears at the end of the member. Additional influence coefficients are needed for this purpose. A considerable quantity of valuable information can be derived from tabulated values.[10] However, there are several gaps that remain to be filled. The writer has been evaluating some of the desired quantities, and it is expected that these results will be made available in the near future.

Two relationships that are of extreme usefulness in a study of the dynamics of frameworks are presented herein. The author refers to one of them but does not elaborate on it. It has been proved[13] that Maxwell's law of reciprocity is valid in the steady-state forced vibration of systems subjected to pulsating loads. This relationship may be stated as follows: The generalized dynamic displacement (linear or angular) at a point A of a structure due to a harmonically varying generalized load (force or moment) at point B is equal to the displacement at B due to the same load applied at A. This relationship can be used to prove that the principle of Müller-Breslau is also applicable. According to this principle: The influence line for a dynamic effect (moment or shear) at some point A of a structure due to a harmonically varying load can be obtained as the deflected shape of the structure due to a very small unit dynamic distortion (rotation or deflection) introduced at point A.

The writer cannot agree with the statement made repeatedly in the paper that a continuous system behaves dynamically very nearly as a series of simply supported beams. This statement is valid only for a few particular cases such as some of the illustrative examples considered in the paper. In a continuous system, the restraint exerted at the ends of a span by the members on either side of it may be of such magnitude that the span may essentially behave as a clamped-ended member. In such cases the moments at the supports can be several times larger than the moments below the concentrated pulsating force. The degree of restraint exerted at the ends of a span depends not only on the mass per unit of length, the span length, and the flexural rigidity of cross section of the adjoining members, but also, to a very large extent, on the frequency of the disturbing force. Because the final moments at the joints of a continuous system may be fairly large, in general, the method of distributing end slopes is not superior to the equivalent method of distributing end moments.

In the analysis and design of structures supporting machinery, the determination of the stresses and deformations of the structure are, of course, of considerable importance. However, the fact should not be overlooked that mere knowledge of these stresses and deformations is not sufficient to insure a satisfactory design. Of equal (and probably of much greater) importance is a knowledge of the natural frequencies of the system. It is the writer's opinion

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[13] "The Theory of Sound," by Lord Rayleigh, Vol. 1, Dover Publications, New York, N. Y., 1945, pp. 150–157.

that the first step in the analysis of frames supporting machinery should be to evaluate the natural frequencies of the system and to compare these frequencies with the operating frequencies of the machinery. If the latter are found to be in resonance with, or sufficiently close to, any one of the natural frequencies of the system, the design should be modified. Having eliminated the undesirable possibility of resonance, it would then be necessary to compute the maximum stresses and deformations to ascertain that allowable design limits have not been exceeded.

Acknowledgments.—The writer gratefully acknowledges the benefit that he has derived from discussing problems of the dynamic response of structures with N. M. Newmark, M. ASCE, and L. E. Goodman, A. M. ASCE, both of the University of Illinois, at Urbana. The writer is also indebted to J. G. Sutherland, J. M. ASCE, for calling his attention to Mr. Gaskell's paper.[4]

WILLIAM A. CONWELL,[14] M.ASCE.—An approach to an important problem in the field of structural engineering has been presented by the author. He has done a particularly good job of cataloging for the profession, in one place, the expressions for the forced vibration of a simple beam resulting from periodic concentrated and uniform loads and from periodic end moments (Eqs. 2*a* to 5*b*, inclusive).

However, some clarification seems necessary for the statements made in the first paragraph under the heading, "1. Theory," and in the first paragraph under the heading, "2. General Method of Analysis." There is no question concerning the application of the statements regarding small end moments and the resulting approximate behavior as simple spans, when the simply supported members have equal natural frequencies of vibration. Similarly, there seems to be some justification for the author's statements when the natural frequencies of the members and their location in a structure permit them to vibrate in such a way that nodes will occur and end moments will be small. There may be structures for which these two conditions do not obtain, with the result that end moments of sufficient magnitude will develop and vibrations of the members will have characteristics other than those for simple beams. Furthermore, even when the second condition holds, the mode of vibration of the structure will not necessarily correspond to that which gives maximum deflections. Larger amplitudes are characteristic of structural vibrations in which the number of nodes is a minimum.

The author states, in the "Synopsis," that "Analyses have been made of the forced vibrations of simple systems—that is, a spring supporting a mass ***." He might have added that valuable lessons may be learned from the study of these simple systems. Such study is indispensable to the practicing engineer applying judgment in the design and analysis of continuous frames. At the risk of being elementary, the writer would like to report the results of one such study of a simple system.

In Fig. 10 is shown a system consisting of two "members." The first member with a weight $W_1 = 1$ lb and a spring with constant $K_1 = 4{,}419$ lb per

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ft would, if supported rigidly at the base of its spring, have a frequency, f_1, equal to 60 cycles per sec for its one mode of free vibration. The second member with a weight $W_2 = 2$ lb and a spring with constant $K_2 = 35{,}353$ lb per ft would, alone, have a frequency, f_2, equal to 120 cycles per sec for its one mode of free vibration. When the system as shown is acted upon by a periodic force $P = 100 \sin 2\pi$ ft resonance will occur, not at the natural frequency of vibration of the individual members (60 cycles per sec and 120 cycles per sec), but at a value of f_1 equal to 55.9 cycles per sec, and a value of f_2 equal to 129.1 cycles per sec. These latter figures represent the frequencies of the first and second principal modes of the free vibration of the system. The effects of the individual members are shown in the results, but they are effects only. The principal contributor to the system's first principal mode of vibration is the first member. Its support is less rigid than when it is considered individually, so it tends to vibrate freely at a lower frequency—55.9 cycles per sec. The second member, with a spring above and below it, is more rigidly supported than when it is considered alone and, therefore, its contribution is to the second, higher frequency of 129.1 cycles per sec.

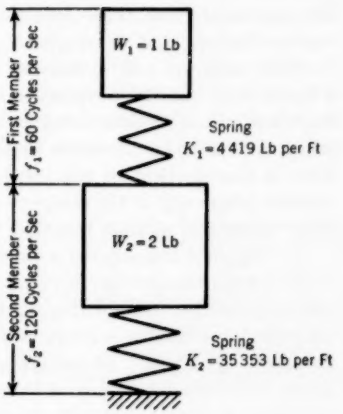
First Member
f_1 = 60 Cycles per Sec
W_1 = 1 Lb
Spring
K_1 = 4 419 Lb per Ft
Second Member
f_2 = 120 Cycles per Sec
W_2 = 2 Lb
Spring
K_2 = 35 353 Lb per Ft

FIG. 10

A similar phenomenon occurs in the study of the vibration of continuous beams. In Fig. 4(*a*) the author shows a two-member system, the first mode of vibration of which will be that shown, with $f_1 = 4$, the natural frequency of vibration of the individual members. In Fig. 5(*a*) there is shown a mode of vibration of the two-beam system corresponding to the natural frequency $f_1 = 4$ in the left member and in each half of the right member, where a node occurs. This mode of vibration is actually the second principal mode of vibration of the two-beam system. The first mode of vibration is characterized by a nodeless span with correspondingly higher deflections than in the second mode, end moments in both spans at the center support, and a lower natural frequency of vibration. Could the closing discussion include an analysis of this first mode of vibration which is usually the most important to the engineer?

There is an implication in Figs. 5, 6, and 7 that the behavior of one member of a vibrating system may be determined if one knows its natural frequency of vibration and that of the other members of the system. It is evident from Eq. 8 that a natural frequency of vibration of 1 cycle per sec for the right-hand span in Fig 5(*a*), treated as a simple span, may be obtianed by taking its span as twice that of the left beam and using m and I with the same values as the left span. An approximation to the natural frequency of the first mode of vibration of a system so constructed is 1.29 cycles per sec. If, however, one

chooses to establish a natural frequency for the right span, simply supported, of 1 cycle per sec by holding the span and mass the same as the left span while taking $I_R = I_L/16$, the natural frequency of the system becomes, roughly, 1.57 cycles per sec. Here, the subscripts R and L indicate the right-hand span and left-hand spans, respectively, of a two-span continuous beam. It must be concluded from these results that the behavior is not independent of the various factors that determine the natural frequency.

The value of 1.57 cycles per sec is approximately the natural frequency of a beam that has the properties of the right-hand span of Fig. 5(*a*) but that is fully restrained against rotation at one support. The value of 1.29 cycles per sec is indicative of the lower relative stiffness of the right-hand beam when its span is double that of the left-hand span. This frequency lies between the natural frequency of the simply supported beam and that for which one end is fully restrained against rotation.

In Fig. 5(*b*) is analyzed a continuous two-span beam in which the members, when simply supported, have natural frequencies of 4 cycles per sec on the left and 5 cycles per sec on the right. The paper deals with the case in which a periodic force having a frequency of 4 cycles per sec is applied to the left-hand member. It should be recognized, however, that maximum deflections will occur when the frequency of the applied force equals the frequency of the first principal mode of free vibration of the two-beam system, which is roughly 4.41 cycles per sec ($m_L = m_R$; $I_L = I_R$; and $L_R = 0.894\ L_L$). Would it not be appropriate for the author to make some reference to the methods for determining natural frequencies of frames as a means for finding the frequency of the periodic load which will produce maximum deflections?

In posing these questions, the writer's only purpose is to clarify parts of the paper that may be puzzling to readers. If, in the closure or in the other discussions, the questions are answered, the author can be credited with having brought into sharp focus an important problem in dynamics. The civil engineer, faced with the design of supports for expensive, high-speed machinery and equipment, is becoming increasingly interested in problems of this type.

C. T. Looney,[16] A.M. ASCE.—The important and interesting problem of the behavior of structural frames subjected to a steady-state forced vibration has been introduced by Mr. Conwell. This behavior is dependent upon the natural frequencies and modes of vibration of the structure, and the method of analysis presented in the paper can be used to determine these natural frequencies and modes of vibration.

To use the writer's method of investigation, it is necessary to introduce a hinge at the end of a member in the structure. An analysis is then made of the end slopes adjacent to the hinge due to equal and opposite periodic end moments. The end slopes will be equal when the frequency of the end moments is equal to a natural frequency of the structure. The natural frequencies of the structure are approximate in value to the natural frequencies of the individual members. The frequencies of the structure are found by a trial-

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and-error method, using appropriate curves. These curves relate a (a measure of the end slopes due to a periodic end moment of a simply supported beam) to the ratio of the frequency of the end moment to the fundamental frequency of the beam. The value of a is obtained from the equation,

$$\theta|_{x=0} = a\frac{M\,L}{E\,I}\sin 2\,\pi\,f\,t \qquad (40)$$

Such a set of curves is identified by the letter A in Fig. 11. The curves in Fig. 11 were obtained by use of Eq. 5b, with α_n equal to zero. Therefore, the damping effect on the natural frequency has been neglected. With the aid of Fig. 11, the natural frequencies of a continuous beam of two spans will be investigated.

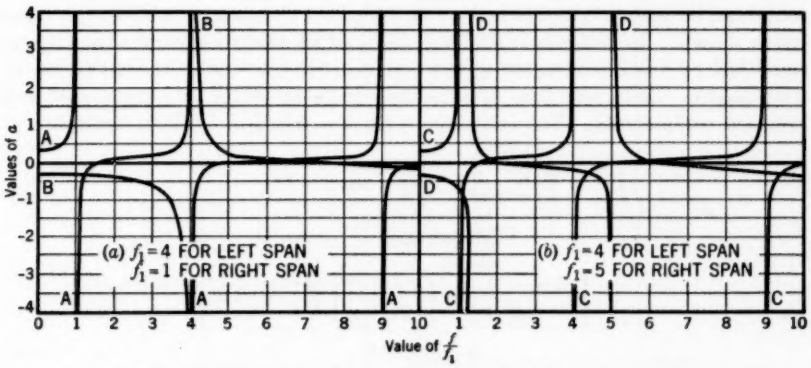

FIG. 11.—THE RELATIONSHIP BETWEEN END SLOPES AND THE RATIO OF THE FREQUENCY OF THE END MOMENT TO THE FUNDAMENTAL FREQUENCY OF THE BEAM

When the two spans have the same frequency, as in Fig. 4(a), the structure will have natural frequencies corresponding to $\frac{f}{f_1} = 1,\ 4,\ 9,\ \ldots,$ equal to those of the individual members. There are also natural frequencies at points where $\theta = 0$, for which $\frac{f}{f_1} = 1.57,\ 5.07,\ \ldots,$ as shown in Fig. 11(a). At these frequencies, the two members act as if their ends were fixed at the center support. For both spans, m, L, E, and I can be varied in any way since, as long as their fundamental frequencies are equal, the natural frequencies of the structure will remain the same.

In Fig. 5(a), $f_1 = 4$ for the left span and $f_1 = 1$ for the right span. In Fig. 11(a), the curves A represent the slope at the center support of the right span due to a moment equal to $M \sin 2\,\pi\,f\,t$, and the curves B represent the slope at the center support of the left span due to a moment equal to $-M \sin 2\,\pi\,f\,t$. The curves B are equal to the curves A with the horizontal scale of B four times greater than that of A. The curves are inverted with respect to each other because of the negative moment. The horizontal scale $\frac{f}{f_1}$ is drawn

with reference to the fundamental frequency of the right span, which in this case is equal to one. The vertical scale for both sets of curves is the same because $E\,I/l$ is the same for both spans. Where these curves intersect, the end slopes, due to an equal and opposite periodic moment at the center support, will be equal. These values of $\frac{f}{f_1}$ are approximately equal to 1.2, 6.4, and 9.6, and, since $f_1 = 1$, these are natural frequencies of the structure. These values of $\frac{f}{f_1}$ are not independent of l, E, and I which determine the vertical scale. In addition, there are natural frequencies for the structure when the natural frequencies of the members coincide. This occurs at $\frac{f}{f_1} = 4, 16, 36, \ldots$.

An accurate determination of the first natural frequency is 1.189. This is obtained by interpolation between the computed values from which the curves were drawn. In order to make the analysis at this frequency, Eq. 5*b* was modified by the introduction of the equation for the sine of the difference of two angles. This was done in order to resolve the slope into horizontal and vertical components. The end slope, with $x = 0$ in Eq. 5*b*, then becomes

$$\theta|_{x=0} = \underbrace{\left[\frac{1}{3} + \frac{2}{\pi^2}\sum_n \frac{1}{n^2}\gamma'_n\right]\frac{M\,l}{E\,I}\sin 2\,\pi\,f\,t}_{\text{(Horizontal)}} - \underbrace{\left[\frac{2}{\pi^2}\sum_n \frac{1}{n^2}\beta^2_n\left(2\,\frac{\delta}{f_n}\,\frac{f}{f_n}\right)\right]\frac{M\,l}{E\,I}\cos 2\,\pi\,f\,t}_{\text{(Vertical)}} \quad \ldots\ldots(41a)$$

in which

$$\beta^2_n\left(1 - \frac{f^2}{f^2_n}\right) = 1 + \gamma'_n \quad \ldots\ldots(41b)$$

and $n = 1, 2, 3, \ldots$.

For the left span, $f_1 = 4$, $f = 1.189$, and $\delta = 0.05$, from which Eq. 41*a* becomes

$$\theta|_{x=0} = \frac{M\,l}{E\,I}(0.353 \sin 2\,\pi\,f\,t - 0.001838 \cos 2\,\pi\,f\,t) \quad \ldots\ldots(42a)$$

For the right span, $f_1 = 1$, $f = 1.189$, and $\delta = 0.05$, and Eq. 41*a* becomes

$$\theta|_{x=0} = \frac{M\,l}{E\,I}(-0.316 \sin 2\,\pi\,f\,t - 0.1305 \cos 2\,\pi\,f\,t) \quad \ldots\ldots(42b)$$

The end slopes $0.353 \sin 2\,\pi\,f\,t$ and $-0.316 \sin 2\,\pi\,f\,t$ were made equal by an additional moment of $-0.279\,M \cos 2\,\pi\,f\,t$. Eqs. 42 then become

$$\theta|_{x=0} = \frac{M\,l}{E\,I}(0.353 \sin 2\,\pi\,f\,t - 0.1003 \cos 2\,\pi\,f\,t) \quad \ldots\ldots(43a)$$

and

$$\theta|_{x=0} = \frac{M\,l}{E\,I}(-0.353 \sin 2\,\pi\,f\,t - 0.0423 \cos 2\,\pi\,f\,t) \quad \ldots\ldots(43b)$$

From Eqs. 43, the stiffness and carry-over slopes were obtained, and used in the analysis in Fig. 12. For the determination of the final moments, the end moment has two components. For the left span, $S\,1 = /0.1003 = 9.97$ as

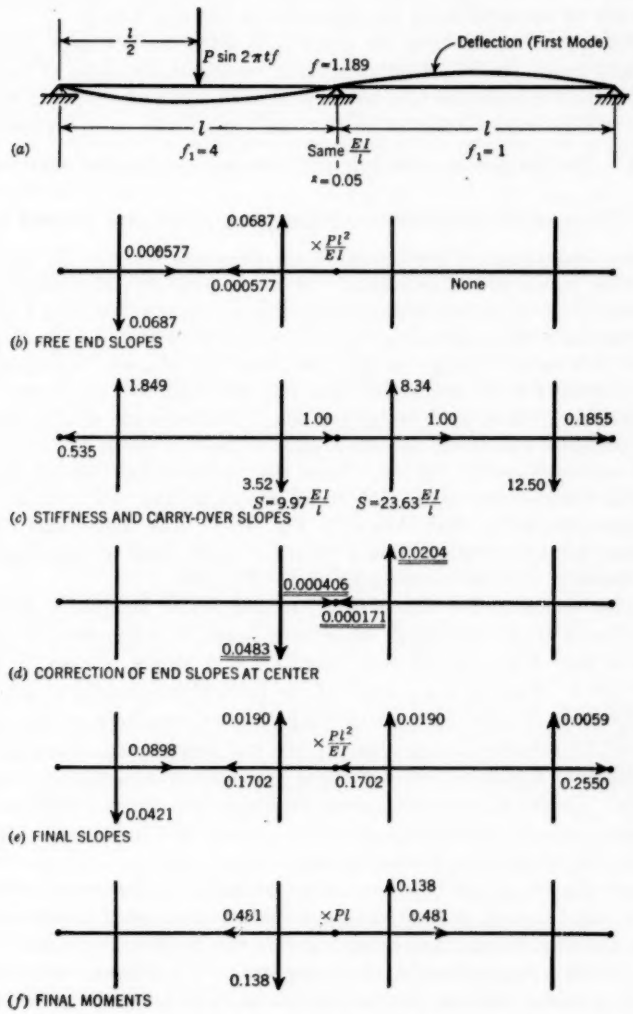

FIG. 12.—END MOMENTS AND SLOPES FOR TWO CONTINUOUS SPANS

shown and $S = 0.279/0.1003 = 2.78$. For the right span, $S = 1/0.0423 = 23.63$ as shown and $S = 0.279/0.0423 = 6.60$. A comparison of the final slopes with the slopes in Fig. 4 indicates that the new slopes are approximately

0.13. A comparison of values of the final moments with the moments in Fig. 4 illustrates that the new moments are 3.1 times greater than those in Fig. 4. The deflection obtained in Fig. 12 is similar to that in Fig. 4 and is proportional to the end slopes; however, neither the deflection nor the moments are of the same order of magnitude as those in Fig. 4.

Fig. 5(*b*) shows an analysis for which $f_1 = 4$ for the left span and $f_1 = 5$ for the right span. In Fig. 11(*b*), curves C represent the slope at the center support for the left span due to a moment $M \sin 2\pi f t$ and curves D represent the slope at the center support of the right span due to a moment $-M \sin 2\pi f t$. The horizontal scale $\frac{f}{f_1}$ is with reference to the left span in which $f_1 = 4$. The points of intersection occur at $\frac{f}{f_1} = 1.1$, 1.7, 4.3, 6.1, and 9.4, and the natural frequencies of the structure are therefore 4.4, 6.8, 17.2, 24.4, and 37.6. These values are approximate. A more accurate determination from the tabulated values would be essential for an accurate solution of a problem.

The purpose of the analysis in Fig. 5(*b*) was to show the effect on the forced vibration of a small change in the frequency of adjacent members. Mr. Conwell is correct in his statement that the deflection will be greater at the natural frequency than at 4 cycles per sec. The estimate of this frequency as being equal to 4.41 cycles per sec is close to the 4.4 cycles per sec obtained from the curves shown in Fig. 11. Since the fundamental frequencies of the two members are almost equal, the beam shown in Fig. 5(*b*) should behave in a manner similar to that shown in Fig. 4(*a*). The deflections due to a steady-state forced vibration should be of the same order of magnitude, with a larger moment over the center support for Fig. 5(*b*).

In order to determine the natural frequencies of structures more complicated than a simple two-span continuous beam, it is necessary to have, in addition to the curves for the end slope at $x = 0$, similar curves for the end slope at $x = l$. Since it is known that the natural frequencies of a structure are approximate in value to those of the individual members, several analyses at different frequencies, using values of the end slopes taken from the curves for the different members, will determine an approximate value for a natural frequency. Tables of accurate values for these end slopes would be helpful for a more accurate determination of the value of the frequency. As a substitute for the tables, Eq. 5*b* may be used.

The peculiarity in the behavior of the examples in the paper, which were noted by Mr. Conwell, is limited to the special cases used, in which $E I/l$ is the same for all members, and limited also to the frequency chosen.

Mr. Masur's suggestion for the computation of stiffness and carry-over factors is a useful addition to the writer's method of analysis. The writer derived the same results by using a finite expression for the deflection of a simply-supported beam due to a periodic end moment, as developed in a paper by R. D. Mindlin, A.M. ASCE, and Mr. Goodman.[16] Values of the functions

[16] "Beam Vibration with Time-Dependent Boundary Conditions," by R. D. Mindlin and L. E. Goodman, *Journal of Applied Mechanics, Paper No. 49—A-19*, ASME.

used were obtained from tables by Keiichi Hayashi.[17] However, these tables require the use of involved methods of interpolation, which outweigh their time-saving value.

The method of analysis described in the paper can be used to determine natural frequencies and modes of vibration for a continuous structure. The analysis for transients is made by combining the steady-state solution with modes of free vibration, so that the deflection and velocity are zero when the time is zero. This analysis can be made by use of simultaneous equations. When the frequency of the forced vibration is equal to or near a natural frequency of the structure, a close approximation can be obtained by combining judiciously the natural modes of vibration with the steady-state solution. This can be done because the shape of the curve for the natural mode of vibration and the steady-state deflection for the same frequency are almost identical.

The references supplied by Mr. Veletsos on the subject of the forced vibration of continuous frames may be of interest to the civil engineering profession. Mr. Veletsos states that "* * * in all previous investigations, the effect of damping on the steady-state forced vibration was assumed to be negligible." Damping has a minor effect on the forced vibrations at frequencies removed from the natural frequencies of the structure; however, the analysis of this problem is of little importance. The important problem for the civil engineer is the analysis of the forced vibrations that occur when synchronism exists between the frequency of the periodic force and the natural frequency of the structure. Analyses of the steady-state forced vibration of continuous structures at synchronism cannot be made without the inclusion of a damping factor. The inclusion of the damping factor and examples of forced vibration at synchronism were covered by the writer.

The writer is familiar with the difficulties that occur in the successive convergence of moment distribution and balancing angle changes. Two familiar examples are the static-load analyses of arches on elastic piers and of a building frame subjected to wind stresses. The realization that an important relationship exists between the method of analysis and the physical behavior of the structure will lead to a discriminating use of the analysis rather than merely automatic application. A discriminating use of the analysis will likewise solve the difficulties of convergence that sometimes occur in the problems described in this paper. In the paper, analyses were made without difficulty, below the natural frequency, at the natural frequency, and above the natural frequency. However, the writer agrees with Mr. Veletsos' statement that tabular values are helpful as an aid in making these computations.

Mr. Veletsos' mention of the writer's statement that "* * * a continuous system behaves dynamically very nearly as a series of simply supported beams" requires comment since this statement was qualified under the heading, "1. Theory," by the words, "In order to attain the greatest deflection due to a periodic force * * *." The properties of the structures and the frequencies used in the examples were chosen so that they behaved in this fashion. If the

[17] "Sieben-und mehratellige Tafeln der Kreis-und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion," by Keiichi Hayashi, Julius Springer, Berlin, Germany, 1926.

properties and frequencies are such that this behavior does not exist, the deflection will not be so great as that encountered. Mr. Veletsos also states that "* * * the method of distributing end slopes is not superior to the equivalent method of distributing end moments." The writer states (under the heading, "8. Summary") that

> "Distribution of the end slopes rather than of the end moments was chosen because initial conditions more closely approximate the final conditions, and therefore there is less correction by distribution."

If initial fixed-end conditions more closely approximated the final conditions, the distribution of end moments would entail less work and would therefore be preferable.

Corrections for Transactions.—The second line following Eq. 2*b* should read "* * * in this case, odd, * * *." Footnote *A* to Table 1, should read "Values of tan α_n are listed * * *." Following Eq. 1, delete the phrase "* * * proportional to the velocity * * *." Table 1, line 4, under the column heading f/f_n, should read "5/20 = $\frac{1}{4}$." In Eq. 27, θ_3 should be equal to $+ 0.5 \frac{P l^2}{E I}$. On page 12, the sentence introducing Eq. 29 should read "Eq. 5*a* * * *". On page 13, in the first of the terms containing β_1, the expression "$\frac{\sin \pi x}{l}$" should read "$\sin \frac{\pi x}{l}$."